

Symmetry between quasielectrons and quasiholes for fractional quantum Hall models defined on lattices

Anne E. B. Nielsen,^{1,2,3} Ivan Glasser,² and Iván D. Rodríguez²

¹*Max-Planck-Institut für Physik komplexer Systeme, D-01187 Dresden, Germany*

²*Max-Planck-Institut für Quantenoptik, D-85748 Garching, Germany*

³*Department of Physics and Astronomy, Aarhus University, DK-8000 Aarhus C, Denmark*

Quasielectrons in fractional quantum Hall systems are known to be much harder to describe theoretically than quasiholes. The problem is that one obtains a singularity in the wavefunction if one tries to naively construct the quasielectron as the inverse of the quasihole. Here we demonstrate that the same problem does not arise in lattice fractional quantum Hall models, so that quasielectrons can indeed be obtained as the inverse of quasiholes. Using this significantly simplified description, which can be applied quite generally, we compute braiding properties of quasielectrons and quasiholes and show that the charge distribution of quasielectrons is minus the charge distribution of quasiholes for different models in the disc and the torus geometry. We also derive few-body Hamiltonians, for which various states containing quasielectrons are ground states.

PACS numbers: 05.30.Pr, 73.43.-f, 11.25.Hf

We are used to that physical particles can be either bosons or fermions, but in certain complex quantum many-body systems it is possible to create anyonic quasiparticles with different and very peculiar properties. Anyons can, e.g., have a charge that is only a fraction of the elementary charge, and if two anyons are exchanged, it leads to a change of the wavefunction that is not just multiplication by a plus or a minus sign. Given that it has a huge impact on the properties of a system, whether the constituent particles are bosons or fermions, it is no surprise that anyons are attracting much attention.

Anyons can appear in connection with the fractional quantum Hall (FQH) effect, which has been realized in certain two-dimensional structures at high magnetic field and low temperature. Various FQH states are obtained for different values of the magnetic field, and anyonic quasiholes/quasielectrons with positive/negative charge can be added to these states by slightly increasing/decreasing the magnetic field. From an experimental point of view quasiholes and quasielectrons hence play a very similar role, but nevertheless it has turned out to be much harder to describe quasielectrons theoretically. As explained, e.g., in [1], this is because it is easier to modify the wavefunction to reduce the electron density locally than to increase it due to the Pauli exclusion principle.

One can introduce quasiholes in the wavefunction by inserting flux tubes with positive flux, but if one instead inserts a flux tube with negative flux to get a quasielectron, one gets a singularity in the wavefunction [2, 3]. This has triggered a lot of work to find suitable wavefunctions of quasielectrons [1–12]. Laughlin proposed a quasielectron wavefunction already in his original paper on the Laughlin state [2]. A different wavefunction with a lower variational energy based on the composite fermion approach was later introduced by Jeon and Jain [5]. Quasielectron wavefunctions have also been constructed for Moore-Read states [1, 7–10], and recently it has been

shown how to obtain a state for a Laughlin quasielectron on the torus [12]. In general, however, the obtained quasielectron wavefunctions are significantly more complicated than the corresponding quasihole wavefunctions, which makes it difficult to compute properties of quasielectrons.

In the last years, there has been significant progress for explicitly computing braiding properties of various types of quasiholes [13–19], but the quasielectrons are again more complicated. For the latter it has been shown that the braiding properties are as expected for the composite fermion version of the Laughlin quasielectrons [20, 21], while the situation for Laughlin’s proposal for the quasielectron wavefunction is unclear [22].

Here, we propose to solve the problem of complicated quasielectron wavefunctions and lack of symmetry between quasielectrons and quasiholes by putting the FQH system on a lattice. In a lattice, the electrons can only be placed at a given set of lattice sites rather than in a continuous space, and this means that the singularity does not appear. It is even possible to define lattice models in such a way that there is an exact symmetry between quasielectrons and quasiholes. A consequence of this symmetry is that the charge distribution of a quasielectron is minus the charge distribution of a quasihole. In this construction, the properties of the quasielectrons are not more complicated to compute than those of the quasiholes. In addition, the wavefunctions are in many cases suitable for Monte Carlo simulations, so that systems with hundreds of lattice sites can be studied. We use this below to compute the shape and braiding properties of quasielectrons. We also derive few-body Hamiltonians for which particular states with anyons are ground states. The ideas presented in the present work open up new and interesting possibilities for investigating important properties of quasielectrons even beyond Laughlin quasielectrons.

Let us first investigate the case of lattice Laughlin states with $1/q$, $q \in \mathbb{N}$, particles per flux unit in a disc geometry. Specifically, we consider a 2D lattice with N sites at the positions $(\text{Re}(z_j), \text{Im}(z_j))$ and take the local Hilbert space on site j to be $|n_j\rangle$, where the occupation number n_j can be either 0 (empty site) or 1 (occupied site). We also allow K quasiholes with charges p_k/q , $p_k \in \mathbb{N}$, to be present at the positions $(\text{Re}(w_k), \text{Im}(w_k))$. The family of lattice Laughlin states are then defined as

$$|\psi\rangle_{\vec{p}} = \mathcal{C}^{-1} \sum_{n_1, \dots, n_N} \delta_n \prod_i \chi_{n_i} \prod_{i,j} (w_i - z_j)^{p_i n_j} \times \prod_{i < j} (z_i - z_j)^{q n_i n_j - n_i \eta - n_j \eta} |n_1, \dots, n_N\rangle, \quad (1)$$

where $\vec{p} \equiv (p_1, p_2, \dots, p_K)$, \mathcal{C} is a real normalization constant, δ_n is unity if the number of particles is $\sum_j n_j = (N\eta - \sum_i p_i)/q$ and zero otherwise, $\chi_{n_j} = e^{i\phi_0 + i\phi_j n_j}$ (with $\phi_0, \phi_j \in \mathbb{R}$) are unspecified single particle phase factors, and η is a parameter that determines the number of lattice sites per area. In the following, we scale the lattice such that the area per lattice site is $2\pi\eta$, which corresponds to setting the magnetic length to unity.

The reason why the states (1) are suitably called lattice Laughlin states is that they, for a suitable choice of the phase factors χ_{n_i} , only differ from the normal Laughlin states in the continuum with q odd by restricting both the allowed positions of the particles and the so-called neutralizing background charge to be on the chosen lattice. This can be seen by noting [23] that $|\psi\rangle \propto \sum_{n_1, \dots, n_N} \langle 0 | W_{p_1} \dots W_{p_K} V_{n_1} V_{n_2} \dots V_{n_N} | 0 \rangle |n_1, \dots, n_N\rangle$, where

$$V_{n_j} = \chi_{n_j} : e^{i(qn_j - \eta)\phi(z_j)/\sqrt{q}} :, \quad (2)$$

$$W_{p_k} = : e^{ip_k\phi(w_k)/\sqrt{q}} :, \quad (3)$$

and comparing to the corresponding results for the continuum Laughlin states in [4]. Here $|0\rangle$ is the vacuum state, $: \dots :$ means normal ordering, and $\phi(z_j)$ is the chiral field of a free, massless boson. In addition, numerical studies [23, 24] for $q \leq 6$ show that the topological properties of the states typically remain after introducing the lattice (the square lattice at $\eta = q/2$ and $q \geq 5$ is an exception).

Our claim in the present work is that, as long as η is not too far from $q/2$ (meaning that the lattice filling factor $\sum_i n_i/N$ is not too far from $1/2$ in the absence of anyons), we can obtain wavefunctions for states with quasielectrons simply by taking some or all of the p_i in (1) to be negative integers. We will now justify this statement by computing the charge, density profile, and braiding statistics of the quasielectrons.

First we note that the condition $\sum_j n_j = (N\eta - \sum_i p_i)/q$ coming from the δ_n factor in the wavefunction shows that if we add a quasielectron to the system then the total number of particles in the system increases by

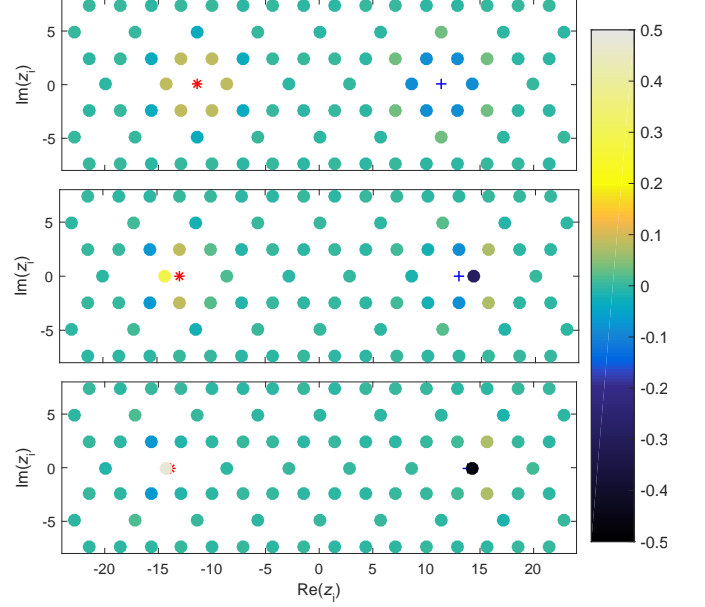


FIG. 1. Modification of the particle density due to the presence of anyons in the lattice Laughlin state (1) at $q = 3$ and half lattice filling ($\eta = q/2$). The lattice is chosen to be a kagome lattice defined on a disc with radius 27.9. A quasielectron (quasihole) with charge $-1/3$ ($+1/3$) is placed at the position $*$ ($+$), and the color of the j th lattice site shows $\langle n_j \rangle_{(-1, +1)} - \langle n_j \rangle_{(0, 0)}$.

$-p_i/q$ independent of η . In the FQH effect the particles are electrons with charge -1 , and the quasielectrons hence have the expected charge p_j/q .

Let us now take $\eta = q/2$ (i.e., lattice filling $\sum_j n_j/N = 1/2 - \sum_i p_i/(Nq)$). In this case, the coefficients of the state (1) are invariant under the transformation $n_i \rightarrow 1 - n_i$ and $p_i \rightarrow -p_i$ up to single particle phase factors (or up to a global phase factor if we choose χ_{n_j} such that $\chi_{n_j} \propto \chi_{1-n_j}$). Note that this transformation transforms quasiholes into quasielectrons and *vice versa*. In particular, it follows that $\langle n_j \rangle_{\vec{p}} = 1 - \langle n_j \rangle_{-\vec{p}}$ and also $\langle n_j \rangle_{\vec{0}} = 1 - \langle n_j \rangle_{\vec{0}}$, so that

$$\langle n_j \rangle_{\vec{p}} - \langle n_j \rangle_{\vec{0}} = -(\langle n_j \rangle_{-\vec{p}} - \langle n_j \rangle_{\vec{0}}) \quad \text{for } \eta = q/2. \quad (4)$$

It is natural to define the density profile of an anyon to be the difference between $\langle n_j \rangle$ when the anyon is present and when the anyon is not present. Equation (4) then says that a quasihole with charge p_i/q has the same density profile as a quasielectron with charge $-p_i/q$ except for a sign. This result is illustrated in Fig. 1 for different positions of the anyons on a kagome lattice. The figure also shows how the shape of the density profile varies with position. We note that even if a quasihole (quasielectron) approaches a lattice site, no singularity occurs. All that happens in that limit is that the probability that the site is empty (occupied by one particle) approaches unity.

What happens if we move away from the symmetric

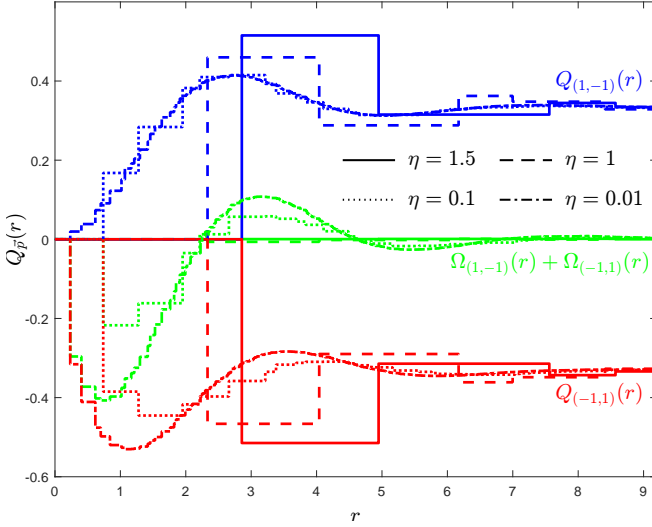


FIG. 2. Excess charge (5) of a quasihole/quaselectron (blue/red) and their sum (green) in the lattice Laughlin state (1) with $q = 3$ for different densities of the lattice sites. The quasihole/quaselectron is placed at the origin at the center of a hexagon in a kagome lattice. The lattice is defined on a disc with radius 27.9 for $\eta = 1.5, 1, 0.1$ and 18.2 for $\eta = 0.01$, and an anyon of the opposite charge is placed at infinity.

point $\eta = q/2$? Figure 2 shows the excess charge

$$Q_{\vec{p}}(r) = - \sum_{\{i \in \{1, 2, \dots, N\} \mid |z_i - w_1| \leq r\}} (\langle n_i \rangle_{\vec{p}} - \langle n_i \rangle_{\vec{0}}) \quad (5)$$

for different values of η , and we observe that the curves for quasiholes and for quaelectrons are close to being symmetric for a broad range of η values. If, however, we let η approach zero (corresponding to that we approach the continuum limit), we observe that the excess charge of the quasihole converges to a fixed curve, while the charge distribution of the quaelectron becomes more and more narrow. This is the singularity in the continuum returning. Figure 2 also shows that the radius of the quasihole is only a bit larger in the lattice than it is in the continuum, which is in line with the results in [19].

The Berry phase $\theta = i \oint_c \langle \psi | \frac{\partial \psi}{\partial w_k} \rangle dw_k + c.c.$ [25, 26] of (1) obtained when moving the k th anyon around a closed curve c evaluates to

$$\theta = i \frac{p_k}{2} \oint_c \sum_i \frac{\langle n_i \rangle_{\vec{p}}}{w_k - z_i} dw_k + c.c., \quad (6)$$

where we have assumed that χ_{n_i} does not depend on w_k . To find the statistics of the anyons we need to compute $\theta_{(w_j \text{ inside})} - \theta_{(w_j \text{ outside})}$, where $\theta_{(w_j \text{ inside})}$ ($\theta_{(w_j \text{ outside})}$) is the Berry phase when the j th anyon is inside (outside) c and not close to c . Our numerical results for $q = 3$ in Figs. 1 and 2 (and for $q = 2$ and $q = 4$, not shown) show that the anyons only alter the particle density in a small region close to w_j . As long as we keep the anyons

well separated, we hence have that $\langle n_i \rangle_{\vec{p}, (w_j \text{ inside})} - \langle n_i \rangle_{\vec{p}, (w_j \text{ outside})}$ is only nonzero close to the two possible positions of the j th anyon and furthermore does not depend on w_k . This allows us to move it outside the integral. Utilizing $\sum_i \text{inside } c (\langle n_i \rangle_{\vec{p}, (w_j \text{ inside})} - \langle n_i \rangle_{\vec{p}, (w_j \text{ outside})}) = -p_j/q$ for w_j not close to c , we arrive at $\theta_{(w_j \text{ inside})} - \theta_{(w_j \text{ outside})} = 2\pi p_j p_k / q$, which is the expected statistics for Laughlin anyons with charges p_j/q and p_k/q .

The mathematics of FQH states defined on a torus is more complicated [27, 28], and it is therefore particularly challenging to construct quasielectrons on the torus. In fact, this problem has been solved only recently for Laughlin quasielectrons in continuous systems [12]. Utilizing the above ideas, however, the construction of quasielectrons in lattice systems on the torus is not more complicated than the construction of quasiholes. We define a torus by specifying the complex numbers r_1 and r_2 and identifying all points in the complex plane separated by $nr_1 + mr_2$ with $n, m \in \mathbb{Z}$. Without loss of generality we shall take r_1 real and assume that $\text{Im}(r_2) > 0$. We define $\tau = r_2/r_1$, $\xi_j = z_j/r_1$, and $\zeta_k = w_k/r_1$. It is then straightforward to generalize the derivation of lattice Laughlin states on the torus in [29] to include quasiholes and quasielectrons:

$$|\psi_l\rangle_{\vec{p}} = \mathcal{C}^{-1} \sum_{n_1, \dots, n_N} \delta_n \prod_i \chi_{n_i} \prod_{i,j} \theta \left[\frac{1/2}{1/2} \right] (\zeta_i - \xi_j, \tau)^{p_i n_j} \times \theta \left[\frac{a + l/q}{b} \right] \left(\sum_{i=1}^N \xi_i (q n_i - \eta) + \sum_{j=1}^K \zeta_j p_j, q\tau \right) \times \prod_{i < j} \theta \left[\frac{1/2}{1/2} \right] (\xi_i - \xi_j, \tau)^{q n_i n_j - n_i \eta - n_j \eta} |n_1, \dots, n_N\rangle. \quad (7)$$

Here, $\theta \left[\frac{a}{b} \right] (\xi, \tau) \equiv \sum_{n \in \mathbb{Z}} e^{i\pi \tau (n+a)^2 + 2\pi i (n+a)(\xi+b)}$ and $l \in \{0, 1, \dots, q-1\}$. For q even $(a, b) = (0, 0)$, and for q odd (a, b) can be either $(0, 0)$, $(0, 1/2)$, $(1/2, 0)$, or $(1/2, 1/2)$. For $\eta = q/2$ and $l = 0$ or $l = q/2$, the derivation leading to Eq. (4) is again valid, and the density profiles for quasielectrons and quasiholes are minus each other. For other values of l , we need to add $l \rightarrow q-l$ to the transformation $n_i \rightarrow 1 - n_i$ and $p_i \rightarrow -p_i$. Since states with different l are connected through spectral flow, and since spectral flow should not alter the density profile of localized anyons, this should, however, not change the picture. In Fig. 3 we show numerically for $q = 2$ that the anyons are localized and obey the expected statistics.

We will now derive few-body Hamiltonians for which some of the above states are ground states. For $\eta = 1$, it can be shown following [23] that the operator

$$\Lambda_i = \sum_{j \neq i} \frac{1}{z_i - z_j} [T_j^{-1} d_j - T_i^{-1} d_i (q n_j - 1)] \quad (8)$$

annihilates the state $|\psi\rangle_{\vec{0}}$, where $T_k = e^{i\phi_k} e^{-i\pi(k-1)}$ and d_k is the bosonic/fermionic annihilation operator acting

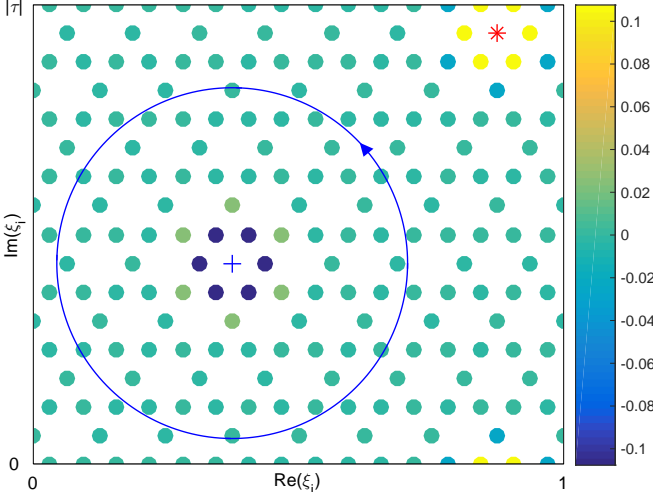


FIG. 3. Quasihole (+) and quasi-electron (*) on the torus in the state $|\psi_0\rangle_{(1,-1)}$ (see (7)) with $q = 2$. The color of the lattice sites shows $\langle n_j \rangle_{(1,-1)} - \langle n_j \rangle_{(0,0)}$. When moving the quasihole around the blue curve we find numerically, using Monte Carlo simulations, that the difference in Berry phase when the quasi-electron is at + and at *, respectively, is $\phi = -3.145$ with a statistical error of order 0.003. This is in agreement with the expected result $-\pi$.

on side k for q even/odd. This enables us to construct a three-body Hamiltonian for this state as $H = \sum_i \Lambda_i^\dagger \Lambda_i$. We find numerically for random choices of the lattice positions that this Hamiltonian has a unique ground state when considering a subspace with a fixed number of particles $\sum_i n_i$.

Let us now add anyons. We first assume $\sum_i p_i = 0$. In this case, the δ_n factor is unaltered, and we have $|\psi\rangle_{\bar{p}} \propto T' |\psi\rangle_{\bar{0}}$, where $T' = \prod_{i,j} (w_i - z_j)^{p_i n_j}$. Therefore $T' \Lambda_i T'^{-1} |\psi\rangle_{\bar{p}} = 0$, and it follows that we obtain a Hamiltonian for the state with anyons simply by redefining $T_k = e^{i\phi_k} e^{-i\pi(k-1)} \prod_i (w_i - z_j)^{p_i}$. Note that this only changes the strengths of the terms in the Hamiltonian.

By moving between 0 and q lattice sites to infinity, we find that the same Hamiltonian also annihilates (1) when $|\sum_i p_i| \leq q/2$ and $\eta = 1$. More generally, by utilizing the method in [24], one can show that the Hamiltonian is valid as long as $-q < \sum_i p_i \leq N$ and $\eta = 1$.

Anyons are often thought of as excitations, but we would like to comment that the above construction with anyons at particular positions in the ground state is actually very convenient for manipulating anyons. If the anyons are excitations, one needs to create anyons and trap them at given positions to be able to move them around in a controlled manner. Here, on the other hand, anyons can be braided by changing the coupling strengths in the Hamiltonian. We can also fuse a quasi-electron and a quasihole to vacuum by bringing them to the same position, and the inverse process would create a quasi-electron-quasihole pair. Note that this works even

when the lattice filling is not $1/2$.

In FQH systems in the continuum, the lowest energy states are holomorphic functions multiplied by a Gaussian. In lattices, however, there is no need to require that the wavefunctions are holomorphic.

In studies of lattice models, it is common to require that the anyons are placed on the lattice sites. Here, we allow the anyons to be at any position in the 2D plane. This picture arises naturally from the Hamiltonians derived above. Another possibility for realizing anyons that are not placed on the lattice sites is to introduce (e.g. in a cold atoms setting) additional particles of a different type to implement the anyons. These particles are bosons or fermions, but start behaving like anyons when interacting in a suitable way with the system. Experiments along such lines have been proposed in [30, 31].

The ideas presented above are quite general and can be applied to other FQH states as well. As an example, we now briefly consider the bosonic Moore-Read state [4]

$$|\psi_{\text{MR}}\rangle \propto \int dZ_1 \cdots \int dZ_M \text{Pf} \left(\frac{1}{Z_i - Z_j} \right) \times \prod_{i < j} (Z_i - Z_j) \prod_j e^{-|Z_j|^2/4} |Z_1, \dots, Z_M\rangle, \quad (9)$$

where Z_j are the positions of the M particles and Pf is the Pfaffian. A lattice version of this state can be obtained in a spin-1 system by replacing (2) with

$$V_{n_j} = \chi(z_j)^{n_j(2-n_j)} : e^{i(n_j-\eta)\phi(z_j)} :, \quad (10)$$

where χ is the field of a majorana fermion and $n_j \in \{0, 1, 2\}$ [32, 33]. As demonstrated in [33], this state reduces to (9) in the continuum limit $\eta \rightarrow 0^+$ up to some unimportant single particle phase factors, and the topological entanglement entropy of the state remains the same for the whole interval $\eta \in [0, 1]$, which suggests that no phase transition occurs when introducing the lattice.

We can obtain wave functions with anyons by inserting operators of the form [4]

$$W_{p_j} = \sigma(w_j) : e^{i \frac{p_j}{2} \phi(w_j)} :, \quad (11)$$

where σ is the spin operator in the Ising model and $p_j = \pm 1$. For the lattice with $\eta = 1$, the transformation $\phi \rightarrow -\phi$ and $n_j \rightarrow 2 - n_j$ transforms the quasiholes ($p_j = +1$) into quasi-electrons ($p_j = -1$) and *vice versa*. In particular, $\langle n_j \rangle_{\bar{p}} = 2 - \langle n_j \rangle_{-\bar{p}}$. The density profiles of quasiholes and quasi-electrons are therefore again symmetric around the mean density of one particle per site.

The presented ideas can also be utilized to obtain information about the topology of continuous systems as long as one can find a path between the continuous system and the lattice system along which no phase transition occurs.

We would like to thank J. Ignacio Cirac and Germán Sierra for discussions. This work has been supported

by the EU Integrated Project SIQS and by the Villum Foundation.

-
- [1] T. H. Hansson, M. Hermanns, and S. Viefers, Phys. Rev. B **80**, 165330 (2009).
 - [2] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
 - [3] R. B. Laughlin, *Elementary theory: The incompressible quantum fluid, in The quantum Hall effect* (Springer, New York, 1987).
 - [4] G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1991).
 - [5] G. S. Jeon and J. K. Jain, Phys. Rev. B **68**, 165346 (2003).
 - [6] T. H. Hansson, C.-C. Chang, J. K. Jain, and S. Viefers, Phys. Rev. B **76**, 075347 (2007).
 - [7] B. A. Bernevig and F. D. M. Haldane, Phys. Rev. Lett. **102**, 066802 (2009).
 - [8] T. H. Hansson, M. Hermanns, N. Regnault, and S. Viefers, Phys. Rev. Lett. **102**, 166805 (2009).
 - [9] G. J. Sreejith, A. Wójs, and J. K. Jain, Phys. Rev. Lett. **107**, 136802 (2011).
 - [10] I. D. Rodriguez, A. Sterdyniak, M. Hermanns, J. K. Slingerland, and N. Regnault, Phys. Rev. B **85**, 035128 (2012).
 - [11] B. Yang and F. D. M. Haldane, Phys. Rev. Lett. **112**, 026804 (2014).
 - [12] M. Greiter, V. Schnells, and R. Thomale, Phys. Rev. B **93**, 245156 (2016).
 - [13] D. Arovas, J. R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. **53**, 722 (1984).
 - [14] B. Paredes, P. Fedichev, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **87**, 010402 (2001).
 - [15] P. Bonderson, V. Gurarie, and C. Nayak, Phys. Rev. B **83**, 075303 (2011).
 - [16] E. Kapit, P. Ginsparg, and E. Mueller, Phys. Rev. Lett. **108**, 066802 (2012).
 - [17] Y.-L. Wu, B. Estienne, N. Regnault, and B. A. Bernevig, Phys. Rev. Lett. **113**, 116801 (2014).
 - [18] A. E. B. Nielsen, Phys. Rev. B **91**, 041106 (2015).
 - [19] Z. Liu, R. N. Bhatt, and N. Regnault, Phys. Rev. B **91**, 045126 (2015).
 - [20] G. S. Jeon, K. L. Graham, and J. K. Jain, Phys. Rev. Lett. **91**, 036801 (2003).
 - [21] G. S. Jeon, K. L. Graham, and J. K. Jain, Phys. Rev. B **70**, 125316 (2004).
 - [22] H. Kjønsberg and J. Myrheim, Int. J. Mod. Phys. A **14**, 537 (1999).
 - [23] H.-H. Tu, A. E. B. Nielsen, J. I. Cirac, and G. Sierra, New J. Phys. **16**, 033025 (2014).
 - [24] I. Glasser, J. I. Cirac, G. Sierra, and A. E. B. Nielsen, arXiv:1609.xxxxx.
 - [25] M. V. Berry, Proc. R. Soc. London, Ser. A **392**, 45 (1984).
 - [26] N. Read, Phys. Rev. B **79**, 045308 (2009).
 - [27] F. D. M. Haldane and E. H. Rezayi, Phys. Rev. B **31**, 2529 (1985).
 - [28] F. D. M. Haldane, Phys. Rev. Lett. **55**, 2095 (1985).
 - [29] A. Deshpande and A. E. B. Nielsen, Journal of Statistical Mechanics: Theory and Experiment **2016**, 013102 (2016).
 - [30] Y. Zhang, G. J. Sreejith, N. D. Gemelke, and J. K. Jain, Phys. Rev. Lett. **113**, 160404 (2014).
 - [31] Y. Zhang, G. J. Sreejith, and J. K. Jain, Phys. Rev. B **92**, 075116 (2015).
 - [32] M. Greiter and R. Thomale, Phys. Rev. Lett. **102**, 207203 (2009).
 - [33] I. Glasser, J. I. Cirac, G. Sierra, and A. E. B. Nielsen, New J. Phys. **17**, 082001 (2015).